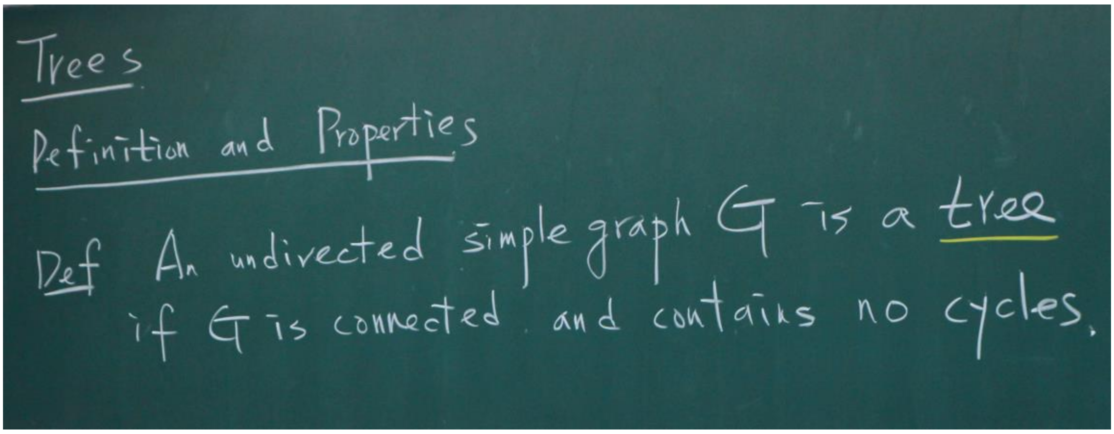
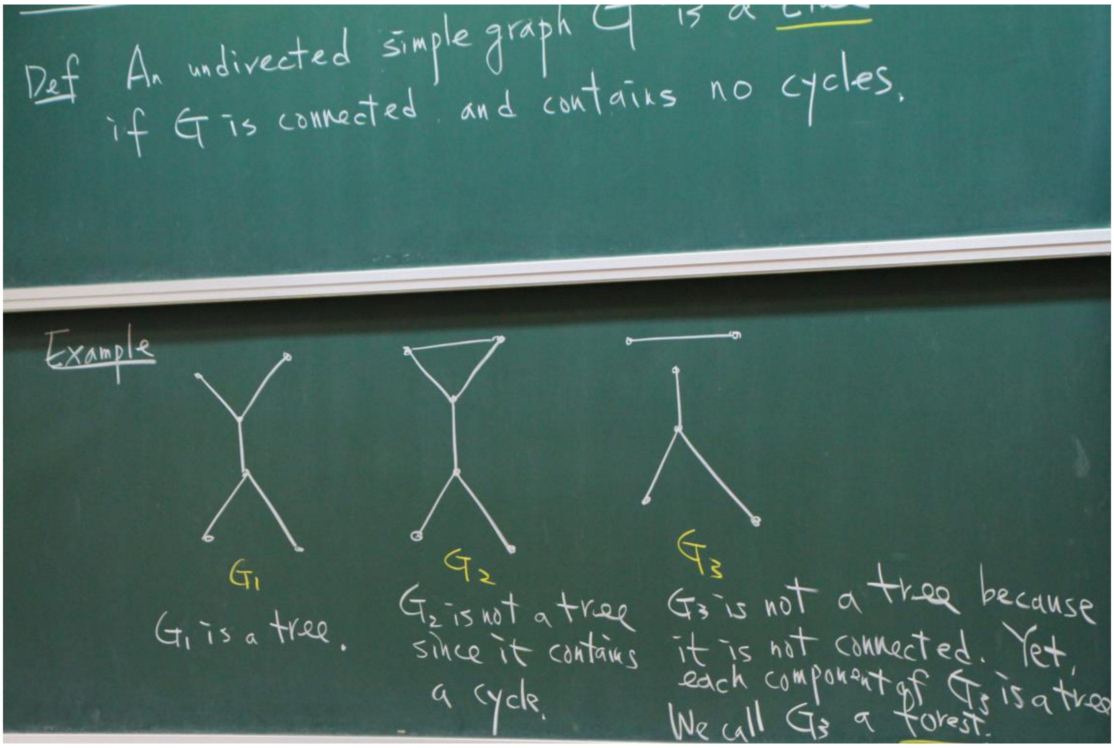
【10920 趙啟超教授離散數學 / 第 23 堂版書】



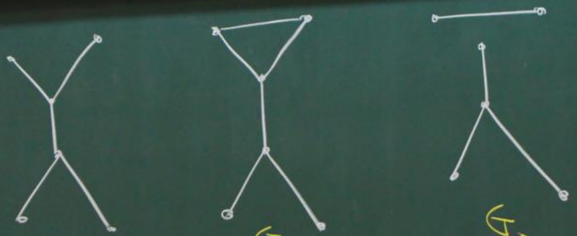


Trees

Definition and Properties

Def An undirected simple graph G is a tree if G is connected and contains no cycles.

Example



G_1 G_2 G_3

G_1 is a tree.

G_2 is not a tree since it contains a cycle.

G_3 is not a tree because it is not connected. Yet, each component of G_3 is a tre[e]

We call G_3 a forest

Property Let $T = (V, E)$ be a tree.

1. If $|V| \geq 2$, there is a unique path connecting any two distinct vertices in T.
2. If $|V| \geq 2$, the graph obtained from T by removing any edge has two components, each of which is a tree.
3. $|V| = |E| + 1$.

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Proof 1. Let a, b be two distinct vertices in T. Since T is connected, there is at least one path in T connecting a and b. If there were more, then from two such paths some of the edges would form a cycle, contradicting the condition that T has no cycles.

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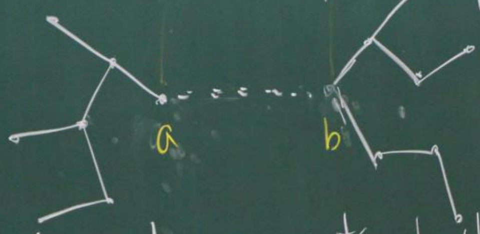
2. Let $\{a, b\}$ be any edge in T and $T' = (V, E')$ be the graph obtained from T after edge $\{a, b\}$ is removed, where $E' = E - \{\{a, b\}\}$. The if T' is connected, there are at least two paths in T connecting a and b, contradicting Property 1. Hence T' is disconnected and the vertices in T' can be partitioned into two

subsets: (1) vertex a and those vertices that can be reached from a by a path in T'; (2) vertex b and those vertices that can be reached from b by a path in T'.



Thus T' has two components, both of which are trees because a cycle in either component would also be in T.

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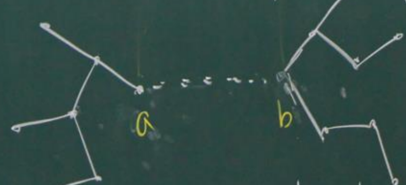

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3.

The proof is by induction on $|V|$.

If $|V| = 1$, then $|E| = 0 = |V| - 1$.

Assume that this true for all trees with k or fewer vertices, where $k \geq 1$.

Now consider a tree $T = (V, E)$ where $|V| = k+1$.

Suppose an edge is removed from T.

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From property 2, we then obtain two trees $T_1 = (V_1, E_1)$ and $T_2 = (V_2, E_2)$ with

$|V_1| + |V_2| = |V| = k+1$ and $|E_1| + |E_2| = |E| - 1$.

Since $|V_1| \geq 1$ and $|V_2| \geq 1$, $|V_1| \leq k$ and $|V_2| \leq k$.

The induction hypothesis applied to T_1 and T_2 gives

$|V_1| = |E_1| + 1$ and $|V_2| = |E_2| + 1$.

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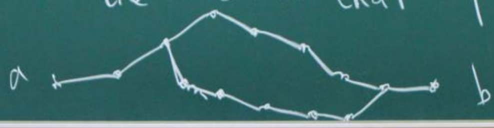


Consequently,

$$|V|=|V_1|+|V_2| = |E_1|+1+|E_2|+1$$
$$=|E|-1+2 = |E|+1.$$

Therefore, this property is true by induction. ■

Theorem

The following statements are equivalent for an undirected simple graph $T=(V,E)$.

(a) T is a tree.

(b) T contains no cycles, and $|V|=|E|+1$.

(c) T is connected, and $|V|=|E|+1$.

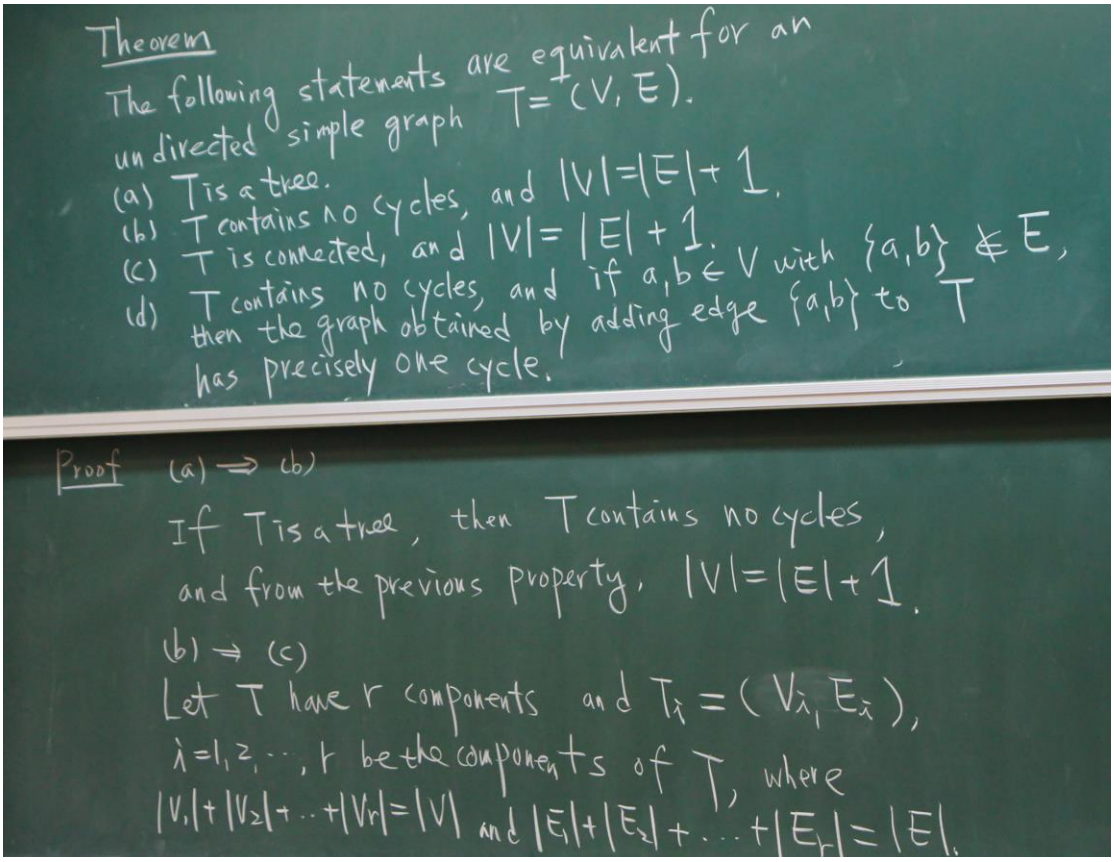
(d) T contains no cycles, and if $a,b\in V$ with $\{a,b\}\notin E$, then the graph obtained by adding edge $\{a,b\}$ to T has precisely one cycle.

Proof (a) $\Rightarrow$ (b)

If T is a tree, then T contains no cycles, and from the previous property, $|V|=|E|+1$.

(b) $\Rightarrow$ (c)

Let T have r components and $T_i=(V_i,E_i)$, $i=1,2,\cdots,r$ be the components of T, where $|V_1|+|V_2|+\cdots+|V_r|=|V|$ and $|E_1|+|E_2|+\cdots+|E_r|=|E|$.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Then each T_i, $i = 1, 2, \dots, r$, is a tree because T_i is connected and contains no cycles (as T contains no cycles). Hence from the previous property
$|V_i| = |E_i| + 1$, for $i = 1, 2, \dots, r$.
Consequently, $|V| = |V_1| + |V_2| + \dots + |V_r| = |E_1| + 1 + |E_2| + 1 + \dots + |E_r| + 1$
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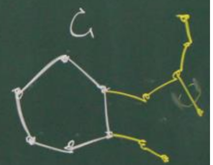
(c) $\Rightarrow$ (d)
Suppose T has a cycle C_1 with t vertices and t edges.
Since T is connected, the remaining vertices of T can each be connected to a vertex in C_1 by a path in T.
Each such connection requires at least one new edge. Consequently, in T, $|E| \geq |V|$ contradicting $|V| = |E| + 1$. So T has no cycles.



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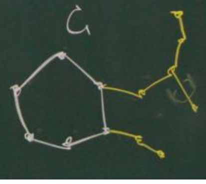
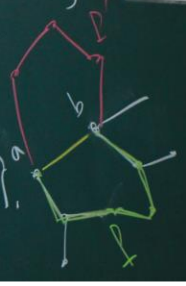


As T is also connected, T is a tree.
Let T' be the graph obtained by adding edge $\{a,b\}$ to T. Since $\{a,b\} \notin E$, there is a unique path P of length at least 2 in T that connects a and b.
In T', $P \cup \{\{a,b\}\}$ is a cycle. If T' contains a second cycle C_1', then C_1' must contain edge $\{a,b\}$.
This second cycle $C_1' = P' \cup \{\{a,b\}\}$ where



P' is a path connecting a and b in T and $P' \neq P$

This contradicting the property that this is a unique path connecting a and b in T.

$(d) \Rightarrow (a)$

If T is not connected, let $T_1 = (V_1, E_1)$, $T_2 = (V_2, E_2)$ be components of T with $a \in V_1$, $b \in V_2$. Then adding the edge $\{a,b\}$ to T would not result in a cycle,

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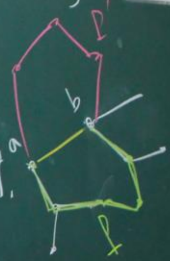

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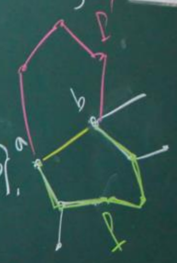

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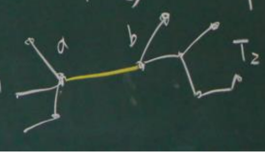
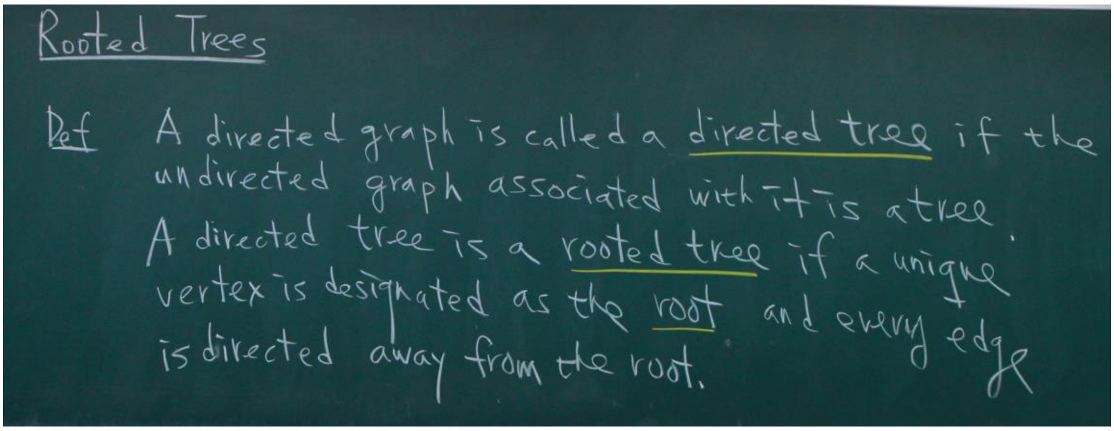


thus contradicting the condition of (d).
Consequently, T is connected. Since T also contains no cycles, T is a tree.

Rooted Trees

Def A directed graph is called a directed tree if the undirected graph associated with it is a tree.
A directed tree is a rooted tree if a unique vertex is designated as the root and every edge is directed away from the root.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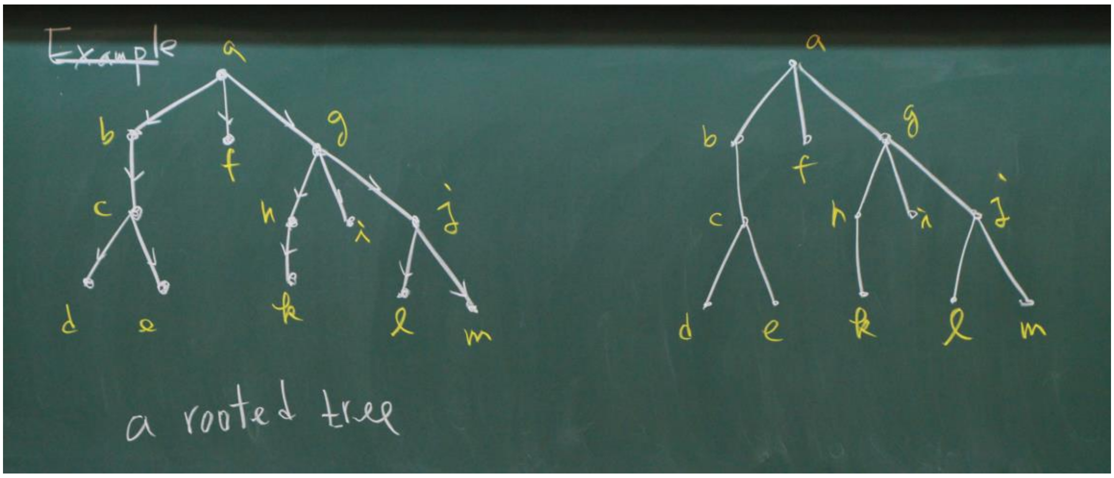
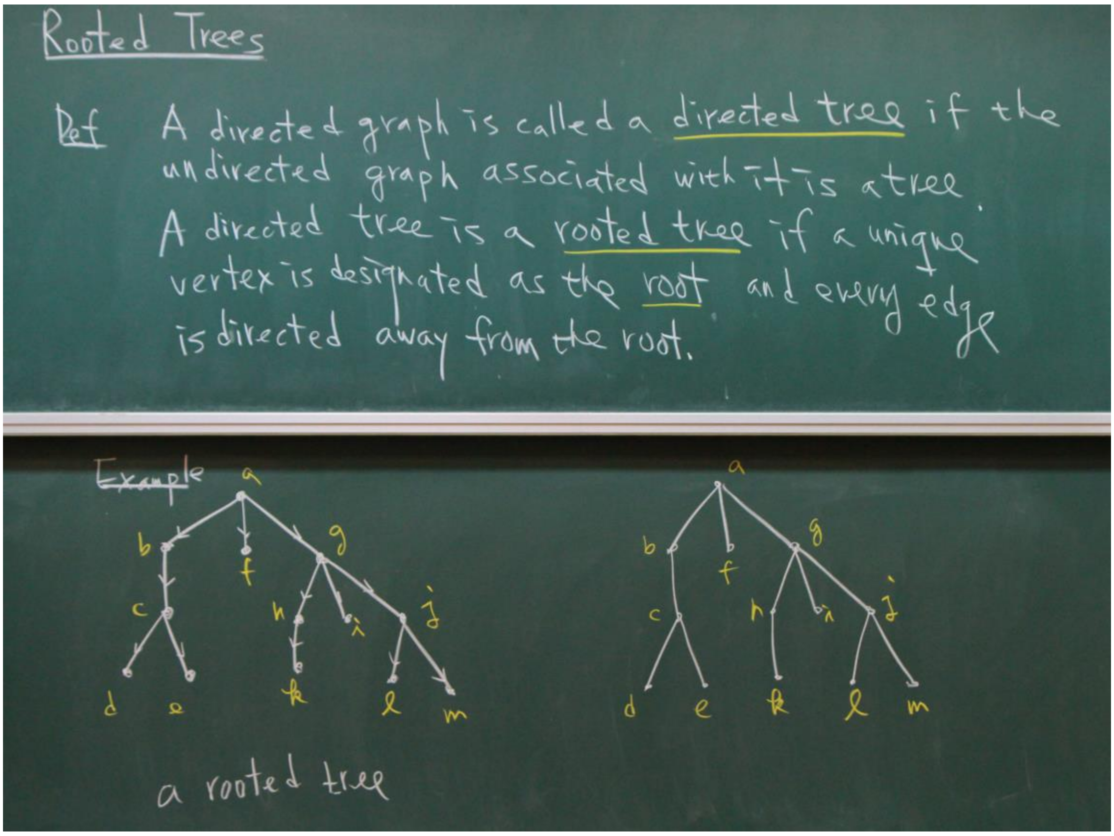

Example
a
b
f
g
c
h
i
j
d
e
k
l
m
a rooted tree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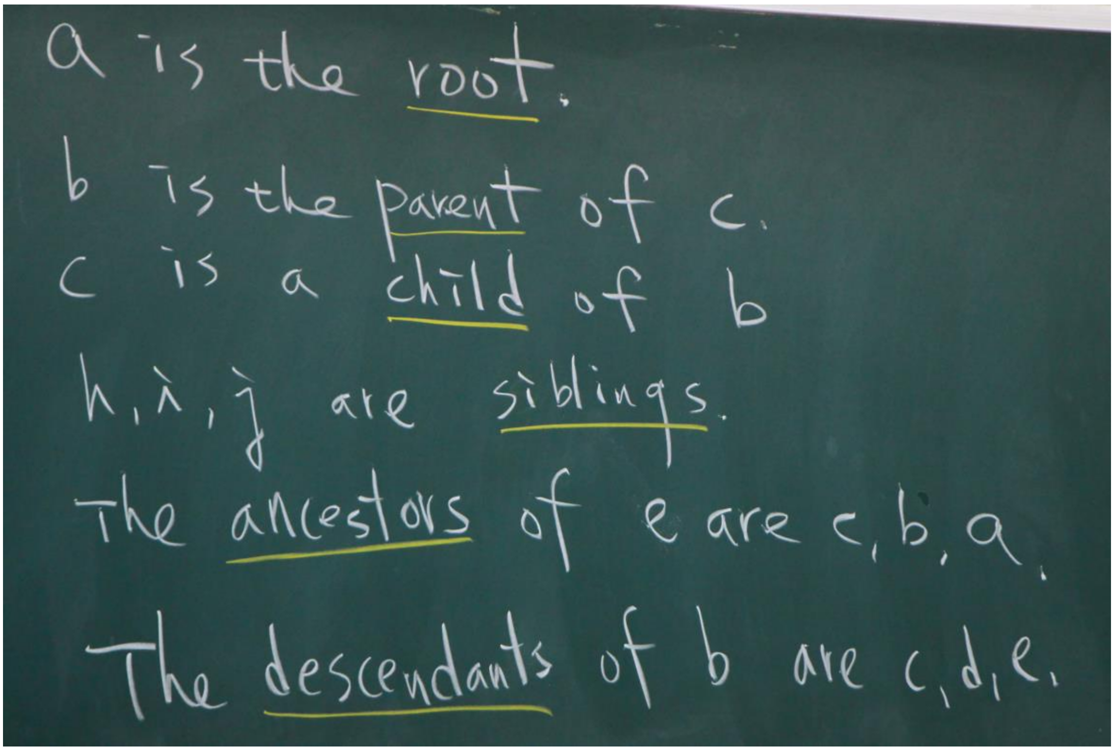
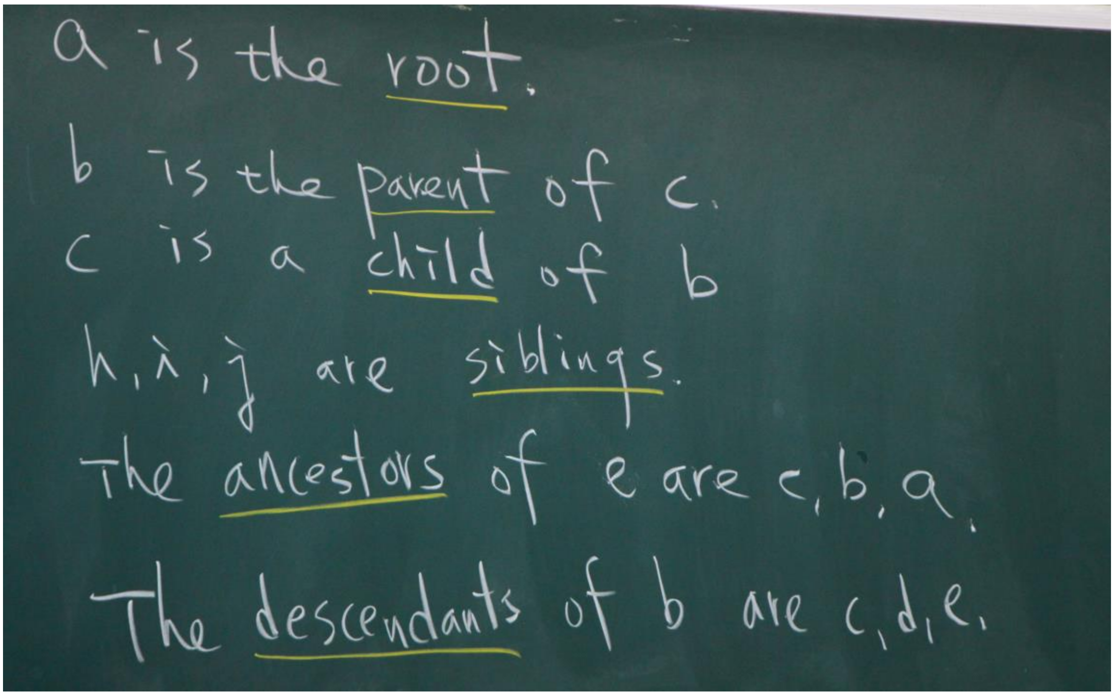

a is the root.
b is the parent of c.
c is a child of b
h, i, j are siblings.
The ancestors of e are c, b, a.
The descendants of b are c, d, e.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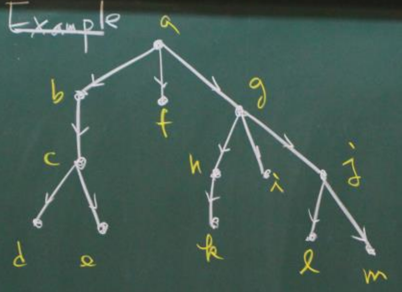
Example



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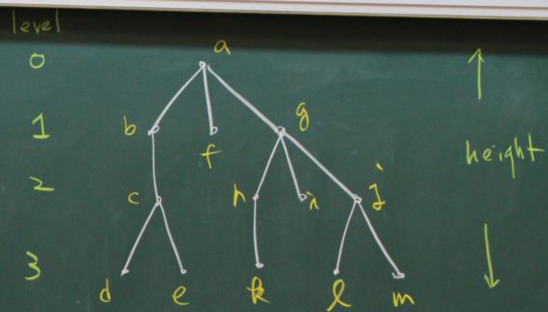
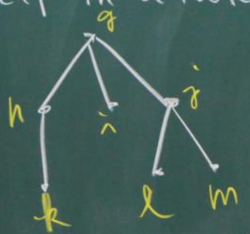


Vertices d, e, f, k, i, l, m are leaves.

Vertices a, b, c, g, h, j are internal vertices.

(A leaf in a rooted tree is a vertex with no children.)



the subtree at g

This rooted tree has height 3.

a is the root.

b is the parent of c.
c is a child of b

h, i, j are siblings.

The ancestors of e are c, b, a.

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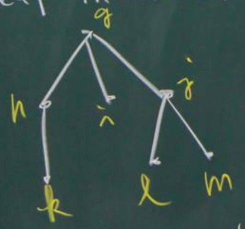

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